Combinatorial Hopf algebras in particle physics IV

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May 29

<u>Slide 2 additional information:</u> "Feynman integrals: special functions and numbers"

$$
\begin{aligned}
-\log(1-z) = \mathrm{Li}_1(z) &= \sum_{n\geq 1} \frac{z^n}{n} \\
\mathrm{Li}_k(z) &= \sum_{n\geq 1} \frac{z^n}{n^k} \\
\Rightarrow \mathrm{Li}_k(1) &= \sum_{n\geq 1} \frac{1}{n^k} = \zeta(k) \qquad \text{MZV}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Li}_2(1) = \zeta(2) &= \frac{\pi^2}{6} \\
\mathrm{Li}_{2n}(1) = \zeta(2n) &\in \mathbb{Q}\cdot\pi^{2n}
\end{aligned}
$$

$$\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3} \notin \mathbb{Q} \qquad \text{it is unknown if it is in } \overline{\mathbb{Q}}$$

MZV-relations:

$$\begin{aligned}\zeta(3)\zeta(5) &= \sum_{\substack{n\geq 1\\ m\geq 1}} \frac{1}{n^3 m^5} \\ &= \sum_{0<n<m} \frac{1}{n^3 m^5} + \sum_{0<m<n} \frac{1}{n^3 m^5} + \sum_{0<n=m} \frac{1}{n^8} \\ &= \zeta(3,5) + \zeta(5,3) + \zeta(8)\end{aligned}$$

Setting $\zeta(3,5) := y_3 y_5$, $\zeta(5,3) := y_5 y_3$;

$$y_5 * y_3 = y_5 y_3 + y_3 y_5 + y_8 \qquad \text{(shuffle product)}$$

<u>Slide 6 additional information:</u> "Integration with hyperlogarithms"

$$\begin{aligned}G(\sigma_1; z) &= \int_0^z \frac{dz_1}{z_1 - \sigma_1} = \log\left(1 - \frac{z}{\sigma_1}\right) \\ G(\sigma_1, \sigma_2; z) &= \int_0^z \frac{dz_1}{z_1 - \sigma_1} G(\sigma_2; z_1)\end{aligned}$$